Problem :

A spherical shell of radius R carries a uniform charge σ_0 on the northern hemisphere and a uniform surface charge $-\sigma_0$ on the southern hemisphere. Find the potential Φ inside and outside the sphere, calculating the coefficients of the series solution explicitly up to A_8 and B_8.

(Based on Griffiths Ch. 3, prob. 21, from the text "Introduction to Electrodynamics" by D.J. Griffiths)

Solution :

Inside and outside the sphere there is no charge, so Laplaces equation applies.

Laplace's equation is separable in 13 coordinate systems. Separable means the solution can be written as a sum of products of functions of the individual coordinates. (cf. "Methods of Theoretical Physics" by Morse and Feshbach, Vol. 1, pp. 655 − 666, for a nice description of these 13 coodinate systems)

While we are free to chose any of these 13 systems in which to expand the solution, it is the boundary conditions of the problem that dictate which coordinate system is the best choice. If this is not clear, it will become so when we write the equations for the boundary conditions on the potential Φ.

Because we have a spherical boundary, (the shell itself) the coordinate system we will use is spherical coordinates. The solution of Laplace's equation in spherical coordinates with azimuthal symmetry (no ϕ dependence) can be expanded in terms of Legendre polynomials :

$$\Phi = \sum_{l=0}^{\infty}\left(A_l\, r^l + \frac{B_l}{r^{l+1}}\right)P_l(\mathrm{Cos}(\theta))$$

For $0 \le r < R$, we must have $B_l = 0$ for $l = 0, 1, 2, 3 \ldots$ so that Φ is finite at when $r = 0$.

i.e.,

$$\Phi_{\mathrm{in}} = \sum_{l=0}^{\infty}\left(A_l\, r^l\right)P_l(\mathrm{Cos}(\theta)) \quad \text{for } 0 \le r < R$$

For $r > R$, we must have $A_l = 0$ for $l = 0, 1, 2, \ldots$

i.e.,

$$\Phi_{\mathrm{out}} = \sum_{l=0}^{\infty}\left(\frac{B_l}{r^{l+1}}\right)P_l(\mathrm{Cos}(\theta)) \quad \text{for } r > R$$

Now, we apply boundary conditions at $r = R$ ...

The potential should be continuous at $r = R$, so $\Phi_{\mathrm{in}}(R) = \Phi_{\mathrm{out}}(R)$, otherwise the electric field, which is the gradient of Φ, will be infinite , which we do not wish to consider delta function systems. Had we choosen a different coordinate system, this continuity equation would have taken a much more complicated form.

i.e.,

$$\sum_{l=0}^{\infty}\left(A_l\, R^l\right)P_l(\mathrm{Cos}(\theta)) \; = \sum_{l=0}^{\infty}\left(\frac{B_l}{R^{l+1}}\right)P_l(\mathrm{Cos}(\theta))$$

so that $A_l R^l = \dfrac{B_l}{R^{l+1}}$, or $B_l = A_l R^{2l+1}$, and so

$$\Phi_{\mathrm{out}} = \sum_{l=0}^{\infty}\left(\frac{A_l\, R^{2l+1}}{r^{l+1}}\right)P_l(\mathrm{Cos}(\theta)) \quad \text{for } r > R$$

Applying the boundary condtion on the normal component of the electric field across a boundary, we get the final boundary condition, which again would be much more complicated in a coordinate system other than spherical coordinates :

$$\partial_r \Phi_{in} - \partial_r \Phi_{out} = \frac{\sigma}{\epsilon_0} \quad \text{at } r = R$$

or,

$$\sum_{l=0}^{\infty} \left(l A_l R^{l-1}\right) P_l(\mathrm{Cos}(\theta)) - \sum_{l=0}^{\infty} \left(\frac{-(l+1) A_l R^{2l+1}}{R^{l+2}}\right) P_l(\mathrm{Cos}(\theta)) = \frac{\sigma}{\epsilon_0}$$

and simplifying gives

$$\sum_{l=0}^{\infty} (2 l + 1) A_l R^{l-1} P_l(\mathrm{Cos}(\theta)) = \frac{\sigma}{\epsilon_0}$$

Now use the fact that the Legendre polynomials are orthogonal functions,

$$\int_{-1}^{1} P_l(x) P_m(x)\, dx = \frac{2}{2 l + 1} \delta_{lm}$$

$\Big($ or, expressing the above orthogonality relation in a more useable form for our problem, after the substitution $x = \cos(\theta)$,

$dx = -\sin(\theta)\, d\theta$,

$$x^{-1}(1) = 0, \quad x^{-1}(-1) = \pi, \quad \text{we get} \int_{-\pi}^{0} P_l(\mathrm{Cos}(\theta)) P_m(\mathrm{Cos}(\theta)) \Big(-\mathrm{Sin}(\theta)\, d\theta\Big) = \int_0^{\pi} P_l(\mathrm{Cos}(\theta)) P_m(\mathrm{Cos}(\theta)) \mathrm{Sin}(\theta)\, d\theta = \frac{2}{2 l + 1} \delta_{lm} \Big)$$

and multiply both sides of equation (1) by $P_m(\mathrm{Cos}(\theta)) \mathrm{Sin}(\theta)$ and integrate :

$$\int_0^{\pi} P_m(\mathrm{Cos}(\theta)) \mathrm{Sin}(\theta)\, d\theta \sum_{l=0}^{\infty} (2 l + 1) A_l R^{l-1} P_l(\mathrm{Cos}(\theta)) = \int_0^{\pi} P_m(\mathrm{Cos}(\theta)) \mathrm{Sin}(\theta) \frac{\sigma}{\epsilon_0}\, d\theta$$

$$\sum_{l=0}^{\infty} (2 l + 1) A_l R^{l-1} \frac{2}{2 l + 1} \delta_{lm} = \int_0^{\pi} P_m(\mathrm{Cos}(\theta)) \mathrm{Sin}(\theta) \frac{\sigma}{\epsilon_0}\, d\theta$$

i.e.,

$$(2 m + 1) A_m R^{m-1} \frac{2}{2 m + 1} = \int_0^{\pi} P_m(\mathrm{Cos}(\theta)) \mathrm{Sin}(\theta)\, \sigma\, d\theta$$

or,

$$A_m = \frac{1}{2 \epsilon_0 R^{m-1}} \left(\int_0^{\pi} P_m(\cos(\theta)) \sin(\theta)\, \sigma\, d\theta\right)$$

i.e.,

$$A_m = \frac{1}{2 \epsilon_0 R^{m-1}} \left(\int_{\pi/2}^{\pi} P_m(\cos(\theta)) \sin(\theta) (-\sigma_0)\, d\theta + \int_0^{\frac{\pi}{2}} P_m(\cos(\theta)) \sin(\theta)\, \sigma_0\, d\theta\right)$$

and with a little help (ok, a lot!) from *Mathematica*, we get

$$A_m = \frac{R^{1-m} \left(\frac{\sqrt{\pi}\, \sigma_0}{2\, \mathrm{Gamma}\left[1-\frac{m}{2}\right] \mathrm{Gamma}\left[\frac{3+m}{2}\right]} - \left(-\frac{\sqrt{\pi}}{2\, \mathrm{Gamma}\left[1-\frac{m}{2}\right] \mathrm{Gamma}\left[\frac{3+m}{2}\right]} + \frac{2\, \mathrm{Sin}[m\, \pi]}{m\, \pi + m^2\, \pi}\right) \sigma_0\right)}{2 \epsilon_0}$$

and

$$B_m = A_m R^{2m+1},$$

$$B_m = \frac{R^{2+m}\left(\frac{\sqrt{\pi}\ \sigma_0}{2\,\text{Gamma}\left[1-\frac{m}{2}\right]\text{Gamma}\left[\frac{3+m}{2}\right]} - \left(-\frac{\sqrt{\pi}}{2\,\text{Gamma}\left[1-\frac{m}{2}\right]\text{Gamma}\left[\frac{3+m}{2}\right]} + \frac{2\,\text{Sin}[m\,\pi]}{m\,\pi+m^2\,\pi}\right)\sigma_0\right)}{2\,\epsilon_0}$$

While the above closed forms for A_m and B_m are nice, it is simpler to evaluate the coefficients as follows :

$$A_0 = \frac{R}{2\,\epsilon_0}\left(\int_{\frac{\pi}{2}}^{\pi} P_0(\cos(\theta))\sin(\theta)(-\sigma_0)\,d\theta + \int_0^{\frac{\pi}{2}} P_0(\cos(\theta))\sin(\theta)\,\sigma_0\,d\theta\right)$$

$$A_0 = \frac{R}{2\,\epsilon_0}(-\sigma_0 + \sigma_0) = 0$$

$$B_0 = 0$$

$$A_1 = \frac{1}{2\,\epsilon_0}\left(\int_{\frac{\pi}{2}}^{\pi} P_1(\cos(\theta))\sin(\theta)(-\sigma_0)\,d\theta + \int_0^{\frac{\pi}{2}} P_1(\cos(\theta))\sin(\theta)\,\sigma_0\,d\theta\right)$$

$$A_1 = \frac{1}{2\,\epsilon_0}(\sigma_0)$$

$$B_1 = \frac{R^3}{2\,\epsilon_0}(\sigma_0)$$

$$A_2 = \frac{1}{2\,\epsilon_0\,R^1}\left(\int_{\frac{\pi}{2}}^{\pi} P_2(\cos(\theta))\sin(\theta)(-\sigma_0)\,d\theta + \int_0^{\frac{\pi}{2}} P_2(\cos(\theta))\sin(\theta)\,\sigma_0\,d\theta\right)$$

$$A_2 = \frac{1}{2\,\epsilon_0\,R^1}(0) = 0$$

$$B_2 = 0$$

$$A_3 = \frac{1}{2\,\epsilon_0\,R^2}\left(\int_{\frac{\pi}{2}}^{\pi} P_3(\cos(\theta))\sin(\theta)(-\sigma_0)\,d\theta + \int_0^{\frac{\pi}{2}} P_3(\cos(\theta))\sin(\theta)\,\sigma_0\,d\theta\right)$$

$$A_3 = \frac{1}{2\,\epsilon_0\,R^2}\left(\frac{-\sigma_0}{4}\right)$$

$$B_3 = \frac{R^5}{2\,\epsilon_0}\left(\frac{-\sigma_0}{4}\right)$$

$$A_4 = \frac{1}{2\,\epsilon_0\,R^3}\left(\int_{\frac{\pi}{2}}^{\pi} P_4(\cos(\theta))\sin(\theta)(-\sigma_0)\,d\theta + \int_0^{\frac{\pi}{2}} P_4(\cos(\theta))\sin(\theta)\,\sigma_0\,d\theta\right)$$

$$A_4 = \frac{1}{2\,\epsilon_0\,R^3}(0) = 0$$

$$B_4 = 0$$

$$A_5 = \frac{1}{2\,\epsilon_0\,R^4}\left(\int_{\frac{\pi}{2}}^{\pi} P_5(\cos(\theta))\sin(\theta)(-\sigma_0)\,d\theta + \int_0^{\frac{\pi}{2}} P_5(\cos(\theta))\sin(\theta)\,\sigma_0\,d\theta\right)$$

$$A_5 = \frac{1}{2\,\epsilon_0\,R^4}\left(\frac{\sigma_0}{8}\right)$$

$$B_5 = \frac{R^7}{2\,\epsilon_0}\left(\frac{\sigma_0}{8}\right)$$

$$A_6 = \frac{1}{2\,\epsilon_0\,R^5}\left(\int_{\frac{\pi}{2}}^{\pi} P_6(\cos(\theta))\,\sin(\theta)\,(-\sigma_0)\,d\theta + \int_0^{\frac{\pi}{2}} P_6(\cos(\theta))\,\sin(\theta)\,\sigma_0\,d\theta\right)$$

$$A_6 = \frac{1}{2\,\epsilon_0\,R^5}(0) = 0$$

$$B_6 = 0$$

$$A_7 = \frac{1}{2\,\epsilon_0\,R^6}\left(\int_{\frac{\pi}{2}}^{\pi} P_7(\cos(\theta))\,\sin(\theta)\,(-\sigma_0)\,d\theta + \int_0^{\frac{\pi}{2}} P_7(\cos(\theta))\,\sin(\theta)\,\sigma_0\,d\theta\right)$$

$$A_7 = \frac{1}{2\,\epsilon_0\,R^6}\left(\frac{-5\,\sigma_0}{64}\right)$$

$$B_7 = \frac{R^9}{2\,\epsilon_0}\left(\frac{-5\,\sigma_0}{64}\right)$$

$$A_8 = \frac{1}{2\,\epsilon_0\,R^7}\left(\int_{\frac{\pi}{2}}^{\pi} P_8(\cos(\theta))\,\sin(\theta)\,(-\sigma_0)\,d\theta + \int_0^{\frac{\pi}{2}} P_8(\cos(\theta))\,\sin(\theta)\,\sigma_0\,d\theta\right)$$

$$A_8 = \frac{1}{2\,\epsilon_0\,R^7}(0) = 0$$

$$B_8 = 0$$

Finally, expanding the first eight terms of the series, we get for $\Phi_{\text{in:}}$

$$\sum_{l=0}^{8}\left(A_l\,r^l\right)P_l\,(\cos\,(\theta)) =$$

$$\frac{r\,\mathrm{Cos}[\theta]\,\sigma_0}{2\,\epsilon_0} - \frac{r^3\left(-3\,\mathrm{Cos}[\theta] + 5\,\mathrm{Cos}[\theta]^3\right)\sigma_0}{16\,R^2\,\epsilon_0} + \frac{r^5\left(15\,\mathrm{Cos}[\theta] - 70\,\mathrm{Cos}[\theta]^3 + 63\,\mathrm{Cos}[\theta]^5\right)\sigma_0}{128\,R^4\,\epsilon_0} - \frac{5\,r^7\left(-35\,\mathrm{Cos}[\theta] + 315\,\mathrm{Cos}[\theta]^3 - 693\,\mathrm{Cos}[\theta]^5 + 429\,\mathrm{Cos}[\theta]^7\right)\sigma_0}{2048\,R^6\,\epsilon_0}$$

and for the first eight terms of Φ_{out} :

$$\sum_{l=0}^{8}\frac{B_l\,P_l(\cos(\theta))}{r^{l+1}} =$$

$$\frac{R^3\,\mathrm{Cos}[\theta]\,\sigma_0}{2\,r^2\,\epsilon_0} - \frac{R^5\left(-3\,\mathrm{Cos}[\theta] + 5\,\mathrm{Cos}[\theta]^3\right)\sigma_0}{16\,r^4\,\epsilon_0} + \frac{R^7\left(15\,\mathrm{Cos}[\theta] - 70\,\mathrm{Cos}[\theta]^3 + 63\,\mathrm{Cos}[\theta]^5\right)\sigma_0}{128\,r^6\,\epsilon_0} -$$

$$\frac{5\,\mathrm{R}^9\left(-35\,\mathrm{Cos}[\theta]+315\,\mathrm{Cos}[\theta]^3-693\,\mathrm{Cos}[\theta]^5+429\,\mathrm{Cos}[\theta]^7\right)\sigma_0}{2048\,\mathrm{r}^8\,\epsilon_0}$$

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